1. 3-Phase Voltages, Currents, and Notation

$$\begin{pmatrix} v_A(t) = V_{A,mag}\cos(\omega t + \phi) \\ v_B(t) = V_{B,mag}\cos(\omega t + \phi + \frac{\frac{2}{3}\pi}{\omega}) \\ v_C(t) = V_{C,mag}\cos(\omega t + \phi + \frac{\frac{4}{3}\pi}{\omega}) \end{pmatrix} = \begin{pmatrix} v_A(t) = V_{A,rms}\sqrt{2}\cos(\omega t + \phi) \\ v_B(t) = V_{B,rms}\sqrt{2}\cos(\omega t + \phi + \frac{\frac{2}{3}\pi}{\omega}) \\ v_C(t) = V_{C,rms}\sqrt{2}\cos(\omega t + \phi + \frac{\frac{4}{3}\pi}{\omega}) \end{pmatrix}$$

$$\xrightarrow{\text{drop } \omega,\sqrt{2}} \begin{pmatrix} v_A(t) = V_A\cos(t + \phi) \\ v_B(t) = V_B\cos(t + \phi + \frac{2}{3}\pi) \\ v_C(t) = V_C\cos(t + \phi + \frac{4}{3}\pi) \end{pmatrix}$$

$$= \begin{pmatrix} \vec{V}_A = V_A \Re e^{j(\phi)} \\ \vec{V}_B = V_B \Re e^{j(\frac{2}{3}\pi + \phi)} \\ \vec{V}_C = V_C \Re e^{j(\frac{4}{3}\pi + \phi)} \end{pmatrix}$$

$$\xrightarrow{\text{Phasor Notation}} \begin{pmatrix} \vec{V}_A = V_A e^{j(\phi)} \\ \vec{V}_B = V_B e^{j(\frac{2}{3}\pi + \phi)} \\ \vec{V}_C = V_C e^{j(\frac{4}{3}\pi + \phi)} \end{pmatrix}$$

$$\xrightarrow{\text{Mag and Angle Notation}} \begin{pmatrix} \vec{V}_A = V_A \angle \phi \\ \vec{V}_B = V_B \angle (\frac{2}{3}\pi + \phi) \\ \vec{V}_C = V_C \angle (\frac{4}{3}\pi + \phi) \end{pmatrix}$$

$$\begin{pmatrix} i_A(t) = I_{A,mag}\cos(\omega t + \theta) \\ i_B(t) = I_{B,mag}\cos(\omega t + \theta + \frac{\frac{2}{3}\pi}{\omega}) \\ i_C(t) = I_{C,mag}\cos(\omega t + \theta + \frac{\frac{4}{3}\pi}{\omega}) \end{pmatrix} = \begin{pmatrix} i_A(t) = I_{A,rms}\sqrt{2}\cos(\omega t + \theta) \\ i_B(t) = I_{B,rms}\sqrt{2}\cos(\omega t + \theta + \frac{2}{3}\pi) \\ i_C(t) = I_{C,rms}\sqrt{2}\cos(\omega t + \theta + \frac{4}{3}\pi) \end{pmatrix}$$

$$\xrightarrow{\text{drop }\omega,\sqrt{2}} \begin{pmatrix} i_A(t) = I_A\cos(t + \theta) \\ i_B(t) = I_B\cos(t + \theta + \frac{2}{3}\pi) \\ i_C(t) = I_C\cos(t + \theta + \frac{4}{3}\pi) \end{pmatrix}$$

$$= \begin{pmatrix} i_A(t) = I_A\Re(e^{j(t+\theta)}) \\ i_B(t) = I_B\Re(e^{j(t+\theta+\frac{2}{3}\pi)}) \\ i_C(t) = I_C\Re(e^{j(t+\theta+\frac{4}{3}\pi)}) \end{pmatrix}$$

$$\xrightarrow{\text{Phasor Notation}} \begin{pmatrix} \vec{I_A} = I_A e^{j\theta} \\ \vec{I_B} = I_B e^{j(\frac{2}{3}\pi+\theta)} \\ \vec{I_C} = I_C e^{j(\frac{4}{3}\pi+\theta)} \end{pmatrix}$$

$$\xrightarrow{\text{Mag and Angle Notation}} \begin{pmatrix} \vec{I_A} = I_A\angle\phi \\ \vec{I_B} = I_B\angle(\frac{2}{3}\pi + \phi) \\ \vec{I_C} = I_C\angle(\frac{4}{3}\pi + \phi) \end{pmatrix}$$

Remark 1. Voltage and current are not necessarily in phase - v_A has phase ϕ and i_A has phase θ

Remark 2. We adopt the vector notation for numbers that are complex, i.e. if $x \in \mathbb{C}$, then we write $\vec{x}$

Remark 3. RMS values are convention, so we adopt the notation: $V_{A,rms} = V_A$

2. 3 Phase Power Is Constant

Recall that power is defined as:

$$s(t) = v(t) \cdot i(t)$$

Recall **Complex Power,** $\vec{S}$, which is just the representation of power, $s(t)$ in the complex domain with phasor notation:

$$\vec{S} = \vec{V}\vec{I}^\star$$

For 3 phase, the power delivered to the load is the sum of the voltages and currents on each line, i.e.:

$$s_{3\phi}(t) = v_A(t)i_A(t) + v_B(t)i_B(t) + v_C(t)i_C(t)$$

Recall the trig identity: $\cos(A)\cos(B) = \frac{1}{2}(\cos(A - B) + \cos(A + B))$:

$$\begin{aligned} s_{3\phi}(t) &= i_A v_A \frac{1}{2}(\cos(\phi-\theta)+\cos(2t+\phi+\theta))+\cdots \\ & i_B v_B \frac{1}{2}(\cos(\phi-\theta)+\cos(2t+\phi+\theta+\frac{4\pi}{3}))+\cdots \\ & i_C v_C \frac{1}{2}(\cos(\phi-\theta)+\cos(2t+\phi+\theta+\frac{8\pi}{3})) \end{aligned}$$

The time-dependent terms are exactly out of phase with each other: (include picture), so they drop out!

We're left with:

$$s_{3\phi}(t) = \frac{1}{2}\cos(\phi-\theta)(i_A v_A + i_B v_B + i_C v_C)$$

Remark 4. There is no time depenedency, the power is constant!

if the line voltages and currents are equal to each other, then:

$$\begin{aligned} s_{3\phi}(t) &= \frac{3}{2} I_{mag} V_{mag} \cos(\phi-\theta) \\ &= 3 I_{rms} V_{rms} \cos(\phi-\theta) \end{aligned}$$

which we can rewrite in several notations:

$$s_{3\phi}(t) = 3IV\cos(\phi-\theta) = 3IV\Re\left(e^{j(\phi-\theta)}\right) \to 3IVe^{j(\phi-\theta)} \to 3IV\angle(\phi-\theta)$$

3. Why 3 phase? Balanced Power Flow vs Single Phase

3.1. **Return Currents.**

Definition 5. A three-phase circuit (3ϕ) is **balanced** if the impedances are equal and the 3 voltage source phasors differ only in their angles, with 120^o angle difference between any pair.

(Circuit diagram)

If we supply 3 loads with 3 independent generators, then we need 3 wires from the generator to the load and 3 return wires back from the load to the generator:

If each load is the equivalent, then each load has the transfer function:

$$\vec{Z} = |Z|\angle Z = Z\angle\zeta$$

So, the currents and voltages supply 3 equivalent loads look like:

$$\begin{aligned} \vec{I_A} &= \frac{\vec{V_A}}{\vec{Z}} = \frac{V_A\angle\theta_A}{Z\angle\zeta} = \frac{V_A}{Z}\angle\theta_A - \zeta \\ \vec{I_B} &= \frac{\vec{V_B}}{\vec{Z}} = \frac{V_B\angle\theta_B}{Z\angle\zeta} = \frac{V_B}{Z}\angle\theta_B - \zeta \\ \vec{I_C} &= \frac{\vec{V_C}}{\vec{Z}} = \frac{V_C\angle\theta_C}{Z\angle\zeta} = \frac{V_C}{Z}\angle\theta_C - \zeta \end{aligned}$$

If we connect all the ends of the loads to the same return wire, then we get:

$$\vec{I}_{return} = \frac{V_A}{Z}\angle(\theta_A-\zeta) + \frac{V_B}{Z}\angle(\theta_B-\zeta) + \frac{V_C}{Z}\angle(\theta_C-\zeta)$$

So, what if the system is balanced? Then the sum of our return currents drops to zero! This eliminates the need for return conductors!

There are additional material savings in other components of the grid (like generators) due to voltage and current cancelation - 3-phase topologies make more efficient use of materials.

3.2. **Line Loss Reduction.** Consider a load that draws a constant amount of power, $\vec{S}$.

Current supplied through a 1 phase topology:

$$\vec{S} = \vec{V}\vec{I}_{1\phi}^{\star}$$

Through a 3-phase topology:

$$\vec{S} = \vec{V}\vec{I}^{\star} = \vec{V}(\vec{I}_A^{\star} + \vec{I_B}^{\star} + \vec{I}_C^{\star})$$

In either topology, we supply the same amount of current:

$$\vec{I}_{1\phi} = \vec{I}_A + \vec{I}_B + \vec{I}_C$$

In a balanced case, all the currents will be equal, thus:

$$\vec{I}_{3\phi} = \vec{I}_A = \vec{I}_B = \vec{I}_C = \frac{\vec{I}_{1\phi}}{3}$$

So, what are the I^2R losses?

For 1 phase topology:

$$P = I_{1\phi}^2 R$$

For a 3 phase topology:

$$\begin{aligned} P &= I_A^2R + I_B^2R + I_C^2R \\ &= 3RI_{3\phi}^2 \\ &= 3R\frac{I_{1\phi}^2}{9} \\ &= \frac{1}{3}RI_{1\phi}^2 \end{aligned}$$

Thus, the line losses are 3 times less for 3 phase than for 1 phase!

3.3. **Voltage regulation with 3 phase rather than 1 phase.** Voltage drop along a feeder:

$$\Delta V = |IZ| = |(I_R + jI_C)(R + jX)| = \sqrt{2I_CI_RRX + I_C^2X^2 + I_R^2R^2(1 + I_C^2X^2)}$$

recall from last time that we can do voltage regulation by injecting complex current (i.e. from a capacitor) so as to reduce I_C. So, let $I_C' = I_C - I_{Reg}$

On a *single-phase* line, we have

$$\Delta V = \sqrt{2(I_C - I_{Reg})I_RRX + (I_C - I_{Reg})^2X^2 + I_R^2R^2(1 + (I_C - I_{Reg})^2X^2)}$$

On a *three-phase* line (assuming the same resistance and reactance as the single phase line - which is not true at all), we have 3 times less current per-line, so we have:

$$\Delta V_{\text{per line}} = \sqrt{2\frac{(I_C - I_{Reg})}{3} I_R R X + \frac{(I_C - I_{Reg})^2}{9} X^2 + I_R^2 R^2 (1 + \frac{(I_C - I_{Reg})^2}{9} X^2)}$$

Plotting these two: